Sheaf Cohomology

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1 Schemes

We recall the basic notions of Schemes. For most of our Algebraic Geometry course we considered finitely generated reduced K-algebras, but we now extend our correspondence between geometry and algebra by including rings. Let A be a ring.

Definition. The spectrum of a ring is the pair (Spec$A$, $\mathcal{O}$), with set of all prime ideals, denoted Spec$A$, and $\mathcal{O}$ a sheaf of rings on Spec$A$. If $\mathfrak{a}$ is an ideal of A then $V(\mathfrak{a}) \subset \text{Spec}A$ is the set of all prime ideals that contain $\mathfrak{a}$

Ideally we'd have a functorial correspondence between a ring A and it's spectrum (Spec$A$, $\mathcal{O}$). But in order to do this we'll need the appropriate category, and it turns out that this is the category of locally ringed spaces.

Definition (Hartshorne pg 72)**.** A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X, and a sheaf of rings $\mathcal{O}_X$ on X, called the structure sheaf of the ringed space. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f : X \to Y$ and a map of sheaves of rings on Y: $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. A ringed space $(X, \mathcal{O}_X)$ is called a locally ringed space if for every $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring. A morphism of locally ringed spaces is a morphism $(f, f^\sharp)$ of ringed spaces such that for each point $p \in X$ the induced map of local rings $f^\sharp_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism of local rings.

Definition (Fibered Product)**.** Let Y be a scheme, and let X be a scheme over Y (i.e. schemes with morphisms to Y). We define the fibered product of X and X over Y, $X \times_Y X$ to be a scheme with morphisms $p_1 : X \times_Y X \to X$ and $p_2 : X \times_Y X \to X$ which make a commutative diagram commute the given morphisms $X \to Y$ such that given any scheme Z over Y, and given morphisms $f : Z \to X$ and $g : Z \to X$

Definition (Separated scheme)**.** Let $f : X \to Y$ be a morphism of schemes. Let $\Delta : X \to X \times_Y X$ for the diagonal morphism.

- The morphism f is called separated if $\Delta(X)$ is a closed subspace of $X \times_Y X$. In this case we say X is separated over Y.
- A scheme X is called separated if it is separated over $\text{Spec}\mathbb{Z}$.

This looks very similar to the condition which yields Hausdorff topological spaces, but we do not always get a Hausdorff topological space is that in the category of schemes, the topological space $X \times X$ is not in general the product of the topological space X with itself (products of schemes aren't products of sets).

Example. If V is a variety over any algebraically closed field k then the associated scheme $t(V)$ is separated over k.

Definition (Noetherian scheme)**.** A scheme is Noetherian if it can be covered by a finite number of open affines $\text{Spec}(A_i)$ each A_i a Noetherian ring.

2 Quasi-Coherent Sheaves

You may find that this is not the definition you find online, a more general one is given for arbitrary ringed spaces we only need the one for schemes.

Notion of sheaf of a module:

It is an unfortunate result that not every sheaf of modules on an affine scheme $\text{Spec}R$ can be associated to an R-module, but if a sheaf is nice enough to have this property than we call this a quasi-coherent sheaf.

Definition. A sheaf of modules $\mathcal{F}$ on a scheme X is called quasi-coherent if there exists an affine open cover $\{U_i\}_{i \in I}$ such that on every $U_i = \text{Spec}R_i$ the restriction of the sheaf $\mathcal{F}\big|_{U_i} \simeq M_i$ is associated to the R_i module M_i

There is the associated notion of a coherent sheaf, which requires that these M_i are finitely generated but we'll not need this.

Example. The structure sheaf $\mathcal{O}_X$ is quasi-coherent on any scheme X since for any open $U = \text{Spec}R \subset X$ the restiction of $\mathcal{O}_X$ to U is isomorphic to R

3 Sheaf Cohomology via Derived Functors

Now review the things we've discussed in class.

Theorem. Given any sheaf S of R-modules there is an injective sheaf I of R-module homomorphisms such that $S \hookrightarrow I$

Which is another way of saying that

Theorem. In the category of R-module sheaves, there are enough injectives.

This theorem gives us an injective resolution from which we build cohomology.

Let X be a topological space, given a sheaf of abelian groups S we can find an injective resolution

$$S \hookrightarrow I_0 \to I_1 \to \cdots$$

Then we define the ith cohomology group as the following quotient:

$$H^i(X, S) = \ker d_i / \text{im } d_{i-1}$$

Where $d_i : I_i \to I_{i+1}$. This $H^i(X, S)$ group is independent of choice of injective resolution of S, and are moreover the right derived functors of the global section functors $\Gamma(X, -)$

This definition of cohomology gives us an immediate result:

Corollary. $H^0(X, S) = \Gamma(X, S)$

Moreover the big theorem says that we can extend short exact sequences of sheaves of abelian groups.

Theorem. Given a short exact sequence of sheaves

$$0 \to S_1 \to S_2 \to S_3 \to 0$$

There exists a long exact sequence of cohomology groups

$$H^0(X, S_1) \to H^0(X, S_2) \to H^0(X, S_3) \to H^1(X, S_1) \to H^1(X, S_2) \to H^1(X, S_3) \to \cdots$$

One result which will prove useful in the next section is the following

Theorem. If $S_{\mathbb{Z}}$ is the constant sheaf over any topology space X then $S_{\mathbb{Z}}(U) = H^0(U, \mathbb{Z})$

We also could relate derived functor cohomology to singular cohomology and deRham cohomology

Theorem.

$$H^i(X, \mathbb{Z}) = H^i_{sing}(X, \mathbb{Z})$$

Grothendieck gave us the following nice result:

Theorem (Vanishing theorem of Grothendieck)**.** Let X be a Noetherian topological space of dim n. Then for all $i > n$ the sheaves of abelian groups $\mathcal{F}$ on X, $H^i(X, \mathcal{F}) = 0$

Proof. Direct limits, see Hartshorne Chpt III.2 □

4 Sheaf Cohomology via Open Covers

The derived functors approach is very nice for theory but in terms of hands on computation it's not useful at all. We now detail an alternative approach to Cohomology of sheaves via open covers call Čech Cohomology.

Let X be a topological space $\mathcal{F}$ a sheaf of abelian groups on X, $\mathcal{U} = (U_i)$ an finite open covering of X. For all $p \in \mathbb{N}$ define the following abelian group:

$$C^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < \ldots < i_p} \mathcal{F}(U_{i_0} \cdots U_{i_p})$$

(We'll suppress the open cover). So an element $\varphi \in C^p(\mathcal{F})$ is a collection of sections in $\mathcal{F}(U_{i_0} \cdots U_{i_p})$, for all intersections of $p+1$ sets taken from teh chosen affine open cover these can be unrelated

For every $p \in \mathbb{N}$ we define a co-boundary map $d^p : C^p(\mathcal{F}) \to C^{p+1}$ as

$$(d^p\varphi)_{i_0,\dots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \varphi_{i_0,\dots,\hat{i_k},..,i_{p+1}}\Big|_{U_{i_0}\cap\cdots\cap U_{i_{p+1}}}$$

With the notation hat meaning i_k is left out. This makes sense since This is the coboundary of $\varphi_{i_0,\dots,\hat{i_k},..,i_{p+1}}$ is a section of $\mathcal{F}$ on $U_{i_0} \cap \cdots \cap \hat{U_{i_k}} \cap \cdots \cap U_{i_{p+1}}$, which contains $U_{i_0} \cap \cdots \cap U_{i_{p+1}}$ as an open subset for all k.

Lemma. For any sheaf of abelian groups $\mathcal{F}$ on a topological space X, the composition $d^{p+1} \circ d^p : C^p(\mathcal{F}) \to C^{p+2}(\mathcal{F})$ is the zero map for all $p \in \mathbb{N}$

Proof. This is successful due to the alternating sum

$$\begin{aligned}
(d^{p+1}(d^p(\varphi)))_{i_0,\dots,i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k (d^p\varphi)_{i_0,\dots,\hat{i_k},..,i_{p+1}} \\
&= \sum_{k=0}^{p+2}\sum_{l=0}^{k-1} (-1)^{k+1} \varphi_{i_0,..,\hat{i_l},..,\hat{i_k},..,i_{p+1}} + \sum_{k=0}^{p+2}\sum_{l=k+1}^{p+2} (-1)^{k+l-1} \varphi_{i_0,..,\hat{i_k},..,\hat{i_l},..,i_{p+1}} \\
&= 0
\end{aligned}$$

□

From this definition of these abelian groups we can form a complex of abelian groups

$$C^0(\mathcal{F}) \to C^1(\mathcal{F}) \to C^2(\mathcal{F}) \to \cdots$$

Each arrow is $d^0, d^1, d^2, etc.$. With this we can define the Cohomology of the complex as the usual $h^p(C^\cdot(\mathcal{U}, \mathcal{F}))$ as $\ker d^p / \text{im } d^{p-1}$. We use this to define Čech cohomology:

Definition. Let X be a topological space and $\mathcal{U}$ an open covering of X. For any sheaf of abelian groups $\mathcal{F}$ on X we define the pth Čech cohomology to be $\check{H}^p(\mathcal{U}, \mathcal{F}) = h^p(C^\cdot(\mathcal{U}, \mathcal{F}))$

Properties of Čech cohomology

Lemma. For any topological space X, open cover $\mathcal{U}$ and sheaf of abelian groups $\mathcal{F}$ then

$$\check{H}^0(\mathcal{U}, \mathcal{F}) \simeq \Gamma(X, \mathcal{F})$$

Lemma. For any sheaf of abelian groups $\mathcal{F}$ on a topological space X, the complex $\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F})$ is a resolution of $\mathcal{F}$, where

$$\mathcal{C}^\cdot(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots i_p} f_* \left(\mathcal{F}|_{U_{i_0}\cap\cdots\cap U_{i_p}} \right)$$

Here $V \subset X$ is open and $f : V \to X$ is inclusion.

Proposition. Let X be a topological space, and let $\mathcal{U}$ be an open covering, and let $\mathcal{F}$ be a flabby sheaf of abelian groups, then for all $p > 0$ $\check{H}(\mathcal{U}, \mathcal{F}) = 0$

Remarks:

- If we fix X a topological space and open cover $\mathcal{U}$ then in general a short exact seq of sheaves of abelian groups on X is not going to always yield a long exact sequence of Čech cohomology groups. As a result it does not always agree with derived functor cohomology, the fix for this is to insist the scheme in question is separated (or at least your space is paracompact).
- Further elaborating on this point about a lack of long exact sequences. It means that $\check{H}^p(\mathcal{U}, -)$ is not a delta functor, Hartshorne gives the example of the trivial covering, a single set and then global section functor $\Gamma(X, -)$ is not exact.
- Its not clear from the definition of Čech cohomology that it does NOT depend on the choice of open cover, which is a large drawback from this definition. It can be shown that the definition of Čech cohomology is independent of choice of cover, but this must be proved.

As mentioned before we do not always get a long exact sequence of Cech cohomology from short exact sequence of sheaves, but there are conditions we can impose on our topological space X: Paracompact Hausdorff spaces, and varieties with the Zariski topology work.

Theorem (Long exact sequences)**.** If $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ is an exact sequence of quasi-coherent sheaves on a variety X then there exists a long exact sequence of vector spaces

$$0 \to H^0(\mathcal{U}, \mathcal{F}_1) \to H^0(\mathcal{U}, \mathcal{F}_2) \to H^0(\mathcal{U}, \mathcal{F}_3) \to H^1(\mathcal{U}, \mathcal{F}_1) \to H^1(\mathcal{U}, \mathcal{F}_2) \to H^1(\mathcal{U}, \mathcal{F}_3) \to \cdots$$

Proof. Consider the commutative diagram of abelian groups.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^p(\mathcal{F}_1) & \longrightarrow & C^p(\mathcal{F}_2) & \longrightarrow & C^p(\mathcal{F}_3) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle d_1^p} & & \downarrow{\scriptstyle d_2^p} & & \downarrow{\scriptstyle d_3^p} & & \\ 0 & \longrightarrow & C^{p+1}(\mathcal{F}_1) & \longrightarrow & C^{p+1}(\mathcal{F}_2) & \longrightarrow & C^{p+1}(\mathcal{F}_3) & \longrightarrow & 0 \end{array}$$

The horizontal maps are induced by the given morphisms of sheaves. We claim the rows are exact. To do this we need 3 results we state without proof:

- Every intersection $U_{i_0} \cap \cdots \cap U_{i_p}$ of affine open subsets from the chosen cover is again an affine variety Spec A.
- From this the three quasi-coherent sheaves $\mathcal{F}_i$ are associated to R-modules M_i for $i = 1, 2, 3$ on this intersection by the following lemma the corresponding sequence of modules is exact

  $$0 \to M_1 \to M_2 \to M_3 \to 0$$

  Lemma. A sequence of R-modules is exact if and only if the sequence of associated sheaves of is exact on X, for X an affine scheme

- This is just a sequence of sections

  $$0 \to \mathcal{F}_1(U) \to \mathcal{F}_2(U) \to \mathcal{F}_3(U) \to 0$$

  By the definition of ——— the rows of the above diagram are just direct products of such sequences so also exact.

The Snake Lemma thus implies there is an induced exact sequence

$$0 \to \ker d_1^p \to \ker d_2^p \to \ker d_3^p \to C^{p+1}(\mathcal{F}_1)/\text{im } d_1^p \to C^{p+1}(\mathcal{F}_2)/\text{im } d_2^p \to C^{p+1}(\mathcal{F}_3)/\text{im } d_3^p \to 0$$

Of kernels and cokernels of the vertical maps in the above diagram. Using the second half $p \to p-1$ and the first half $p \to p+1$ of the sequence separately we can make a new commutative diagram with exact rows. Here Coker $d_1^{p-1} = C^p(\mathcal{F}_1)/\text{im } d_1^{p-1}$

$$\begin{array}{ccccccccc}
 & & \text{Coker } d_1^{p-1} & \longrightarrow & \text{Coker } d_2^{p-1} & \longrightarrow & \text{Coker } d_3^{p-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_1^{p+1} & \longrightarrow & \ker d_2^{p+1} & \longrightarrow & \ker d_3^{p+1} & &
\end{array}$$

Where the vertical maps are induced by d_i^p for $i = 1, 2, 3$. Applying the Snake Lemma again we get an exact sequence of cohomology.

$$\check{H}^p(\mathcal{U}, \mathcal{F}_1) \to \check{H}^p(\mathcal{U}, \mathcal{F}_2) \to \check{H}^p(\mathcal{U}, \mathcal{F}_3) \to \check{H}^{p+1}(\mathcal{U}, \mathcal{F}_1) \to \check{H}^{p+1}(\mathcal{U}, \mathcal{F}_2) \to \check{H}^{p+1}(\mathcal{U}, \mathcal{F}_3)$$

as the kernels and cokernels of the vertical maps in this new diagram are by definition just the cohomology spaces. This theorem is now gotten by combining these exact sequences for all p. □

We can construct the Čech cohomology of the general scheme X via direct limits: If $\mathcal{U} > \mathcal{B}$ are two open covers of X, then we get a homomorphism $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{B}, \mathcal{F})$, such that we get the Čech cohomology of the general scheme:

$$\check{H}^p(X, \mathcal{F}) = \lim_{\to} \check{H}^p(\mathcal{U}, \mathcal{F})$$

Proposition. Let X be an affine scheme and $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$ exact sequence of $\mathcal{O}_X$ modules and assume $\mathcal{F}_1$ is quasi-coherent, then the sequence

$$0 \to \Gamma(X, \mathcal{F}_1) \to \Gamma(X, \mathcal{F}_2) \to \Gamma(X, \mathcal{F}_3) \to 0$$

is exact.

Example. Example of Čech cohomology in action: Of S^1 in its usual topology. Let $\mathbb{Z}$ be the constant sheaf, and $\mathcal{U}$ be the open covering by two connected open semi-circles U, V which overlap at each end so $U \cap V$ is two small intervals. Then

$$C^0 = \Gamma(U, \mathbb{Z}) \times \Gamma(U, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$$

$$C^1 = \Gamma(U \cap V, \mathbb{Z}) = \mathbb{Z} \times \mathbb{Z}$$

(Recall that for long exact sequences of cohomology global sections are H^0)

The map $d : C^0 \to C^1$ sends (a, b) to $(b - a, b - a)$, so $\check{H}^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$, and $\check{H}^0(\mathcal{U}, \mathbb{Z}) = \mathbb{Z}$

5 Comparison of the two

We'd now like to compare the previously build derived functor approach with the current Čech cohomology, and show that they're the same. The first result gives a map between these in the general case, but if we strengthen the ask of what each of $X, \mathcal{F}, \mathcal{U}$ can be we get the desired result.

Lemma (Lemma 4.4 in H Chpt 3)**.** Let X be a topological space, and $\mathcal{U}$ an open covering. Then for all $p \geq 0$ there is a natural transformation

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \to H^p(X, \mathcal{F})$$

Functorial in $\mathcal{F}$

Proof. See Hartshorne chapter 3 page 221. □

If we ask more of these three objects then we obtain an isomorphism:

Theorem. Let X be a noetherian separated scheme, let $\mathcal{U}$ be an open affine cover of X, and $\mathcal{F}$ a quasi-coherent sheaf on X. Then for all $p \geq 0$ the natural maps of the above lemma are isomorphisms:

$$\check{H}^p(\mathcal{U}, \mathcal{F}) \simeq H^p(X, \mathcal{F})$$

We also get Leray's theorem:

Theorem (Leray)**.** Let $\mathcal{F}$ be a sheaf on a topological space X and $\mathcal{U}$ an open cover of X. If $\mathcal{F}$ is acyclic on every finite intersection of elements in the open cover then

$$\check{H}^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$$

Other ways they coincide: If X is a Hausdorff paracompact space, then derived functor cohomology coincdes with Čech cohomology for all sheaves

6 Bibliography

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